1. Compute the Laplace transform (and find an abscissa of convergence) of the shifted Heaviside function $H_a : [0, +\infty) \to \mathbb{R}$,

$$H_a(t) = \begin{cases} 0, & t \leqslant a, \\ 1, & t \geqslant a. \end{cases}$$

2. Compute the Laplace transforms of the modulated hyperbolic trigonometric functions:

$$f(t) = e^{-at} \cosh(\omega t), \quad g(t) = e^{-at} \sinh(\omega t).$$

3. In each of the following cases, find a function $f : [0, +\infty) \to \mathbb{C}$ such that $\mathcal{L}[f](z) = F(z)$. You may use the Laplace transforms of the explicit examples we have computed in class or in previous exercises.

(a) $F(z) = \dfrac{4z}{z^2 + 64}$,

(b) $F(z) = \dfrac{z}{(z+1)(z+2)}$,

(c) $F(z) = \dfrac{1}{z^3 + z}$.

(Hint: You might want to use some algebra to split the above expressions into sums of functions for which you can recognise their Laplace transforms.)

4. Using the residue theorem, compute the inverse Laplace transform of the following functions:

(a) $F(z) = \frac{1}{(z+1)^2(z+2)}$,

(b) $F(z) = \frac{z^2}{(z^2+1)^2}$.

5. Solve the 2^{nd} order initial value problem

$$\begin{cases} y''(x) + 2y'(x) + y(x) = f(x) & \text{for } x > 0, \\ y(0) = 1, \quad y'(0) = 1 \end{cases}$$

in the following two cases:

(a) $f(t) = 0$,

(b) $f(t) = t$.

6. Solve the 3^{rd} order initial value problem

$$\begin{cases} y'''(t) + y'(t) = te^{-t} \quad \text{for } t > 0, \\ y(0) = 0, \quad y'(0) = 0, \quad y''(0) = 1. \end{cases}$$

Solutions

1. Since $|H_a(t)| \leqslant 1$ for all t, we have that 0 is an abscissa of convergence for $\mathcal{L}[H_a]$, since, for any $\gamma > 0$

$$\int_0^{+\infty} |H_a(t)| e^{-\gamma t}\, dt \leqslant \int_0^{+\infty} e^{-\gamma t}\, dt = \frac{1}{\gamma} < +\infty.$$

Thus, computing the Laplace transform of $H_a(t)$, we have for any z with $\mathrm{Re}(z) > 0$:

$$\mathcal{L}\{H_a(t)\}(z) = \int_0^{\infty} H_a(t) e^{-zt}\, dt.$$

Since $H_a(t) = 0$ for $t < a$ and $H_a(t) = 1$ for $t \geqslant a$, the integral simplifies to:

$$\mathcal{L}\{H_a(t)\}(z) = \int_a^{\infty} e^{-zt}\, dt.$$

Compute the integral:

$$\int_a^{\infty} e^{-zt}\, dt = \left[\frac{e^{-zt}}{-z}\right]_a^{\infty} = 0 - \left(\frac{e^{-az}}{-z}\right) = \frac{e^{-az}}{z}.$$

Therefore, the Laplace transform is:

$$\mathcal{L}\{H_a(t)\}(z) = \frac{e^{-az}}{z}.$$

2. Recall the definitions of the hyperbolic functions:

$$\cosh(\omega t) = \frac{e^{\omega t} + e^{-\omega t}}{2}, \quad \sinh(\omega t) = \frac{e^{\omega t} - e^{-\omega t}}{2}.$$

Laplace transform of $f(t) = e^{-at}\cosh(\omega t)$:

$$f(t) = e^{-at} \cdot \frac{e^{\omega t} + e^{-\omega t}}{2} = \frac{1}{2}\left(e^{-(a-\omega)t} + e^{-(a+\omega)t}\right).$$

Using the Laplace transform of an exponential:

$$\mathcal{L}\{e^{-\lambda t}\}(s) = \frac{1}{s+\lambda}, \quad \text{for } \mathrm{Re}(s+\lambda) > 0,$$

we obtain:

$$\mathcal{L}\{f(t)\}(s) = \frac{1}{2}\left(\frac{1}{s+a-\omega} + \frac{1}{s+a+\omega}\right).$$

Combine the terms:

$$\mathcal{L}\{e^{-at}\cosh(\omega t)\}(s) = \frac{s+a}{(s+a)^2-\omega^2}.$$

Laplace transform of $g(t) = e^{-at}\sinh(\omega t)$**:**

$$g(t) = e^{-at}\cdot\frac{e^{\omega t}-e^{-\omega t}}{2} = \frac{1}{2}\left(e^{-(a-\omega)t} - e^{-(a+\omega)t}\right).$$

Then:

$$\mathcal{L}\{g(t)\}(s) = \frac{1}{2}\left(\frac{1}{s+a-\omega} - \frac{1}{s+a+\omega}\right),$$

Combine the terms:

$$\mathcal{L}\{e^{-at}\sinh(\omega t)\}(s) = \frac{\omega}{(s+a)^2-\omega^2}.$$

Final Answers:

$$\boxed{\begin{aligned}\mathcal{L}\{e^{-at}\cosh(\omega t)\}(s) &= \frac{s+a}{(s+a)^2-\omega^2},\\ \mathcal{L}\{e^{-at}\sinh(\omega t)\}(s) &= \frac{\omega}{(s+a)^2-\omega^2},\end{aligned}}$$

valid for $\mathrm{Re}(s+a) > |\omega|$.

3. In each of the following cases, find a function $f : [0,+\infty) \to \mathbb{C}$ such that $\mathcal{L}[f](z) = F(z)$.

(a) $F(z) = \dfrac{4z}{z^2+64}$

Recall that the Laplace transform of $\cos(\omega t)$ is:

$$\mathcal{L}\{\cos(\omega t)\}(z) = \frac{z}{z^2+\omega^2}, \quad \text{for } \mathrm{Re}(z) > 0.$$

This matches our case with $\omega = 8$, so:

$$F(z) = 4\cdot\frac{z}{z^2+8^2} = \mathcal{L}\{4\cos(8t)\}(z).$$

Answer: $f(t) = 4\cos(8t)$.

(b) $F(z) = \dfrac{z}{(z+1)(z+2)}$

Use partial fraction decomposition:

$$\frac{z}{(z+1)(z+2)} = \frac{A}{z+1} + \frac{B}{z+2}.$$

Multiply both sides by $(z+1)(z+2)$:

$$z = A(z+2) + B(z+1).$$

Expanding:

$$z = Az + 2A + Bz + B = (A+B)z + (2A+B).$$

Match coefficients:

$$\begin{cases} A+B=1, \\ 2A+B=0. \end{cases} \quad \Rightarrow A=-1, \quad B=2.$$

So:

$$F(z) = \frac{-1}{z+1} + \frac{2}{z+2}.$$

Use:

$$\mathcal{L}\{e^{-at}\}(z) = \frac{1}{z+a}.$$

Answer: $f(t) = -e^{-t} + 2e^{-2t}$.

(c) $F(z) = \dfrac{1}{z^3+z}$

Factor the denominator:

$$F(z) = \frac{1}{z(z^2+1)}.$$

Use partial fractions:

$$\frac{1}{z(z^2+1)} = \frac{A}{z} + \frac{Bz+C}{z^2+1}.$$

Multiply both sides by $z(z^2+1)$:

$$1 = A(z^2+1) + (Bz+C)(z).$$

Expand:

$$1 = Az^2 + A + Bz^2 + Cz = (A+B)z^2 + Cz + A.$$

Match coefficients:

$$\begin{cases} A+B=0, \\ C=0, \\ A=1. \end{cases} \quad \Rightarrow A=1, \quad B=-1, \quad C=0.$$

So:

$$F(z) = \frac{1}{z} - \frac{z}{z^2+1}.$$

We recognize:

$$\mathcal{L}[1](z) = \frac{1}{z}, \quad \mathcal{L}[\cos(t)](z) = \frac{z}{z^2+1}.$$

Answer: $f(t) = 1 - \cos(t)$.

4. Using the residue theorem, compute the inverse Laplace transform of the following functions:

(a) $F(z) = \dfrac{1}{(z+1)^2(z+2)}$

Let $f(t) = \mathcal{L}^{-1}\{F(z)\}(t)$. Note that the singularities of F are at $z = -1, -2$. Choosing any $\gamma \in \mathbb{R}$ such that $\gamma > \max\{-1, -2\}$ (for instance, $\gamma = 0$ suffices), the formula for the inverse Laplace function reads:

$$f(t) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{zt} \frac{1}{(z+1)^2(z+2)}\, dz.$$

The integrand $e^{zt}F(z)$ has poles at $z = -1$ (order 2) and $z = -2$ (simple pole). We compute the residues of $e^{zt}F(z)$.

At $z = -2$:

$$\mathrm{Res}_{z=-2}\left(\frac{e^{zt}}{(z+1)^2(z+2)}\right) = \lim_{z\to -2} \frac{e^{zt}}{(z+1)^2} = \frac{e^{-2t}}{1^2} = e^{-2t}.$$

At $z = -1$:

$$\begin{aligned}\mathrm{Res}_{z=-1}\left(\frac{e^{zt}}{(z+1)^2(z+2)}\right) &= \frac{d}{dz}\left[\frac{e^{zt}}{z+2}\right]_{z=-1} = \left[\frac{te^{zt}}{z+2} - \frac{e^{zt}}{(z+2)^2}\right]_{z=-1} \\ &= \left[\frac{te^{-t}}{1} - \frac{e^{-t}}{1^2}\right] = e^{-t}(t-1).\end{aligned}$$

In order to compute the integral along $\mathrm{Re}(z) = \gamma$, as we did in class, we close the loop towards the left i.e. we close the line segment $t \to \gamma + it$ for $t \in [-R, R]$ with the half circle $\theta \to Re^{i\theta}$ for $\theta \in [\frac{\pi}{2}, \frac{3\pi}{2}]$ and send $R \to +\infty$.[1] This loop will contain all the poles of $e^{tz}F(z)$ and can be computed using the residue theorem. The integral over the half circle will go to 0 as $R \to +\infty$, leaving us with:

$$\int_{\gamma - i\infty}^{\gamma + i\infty} e^{zt} \frac{1}{(z+1)^2(z+2)}\, dz = 2\pi i\Big(\mathrm{Res}_{z=-1}\left(\frac{e^{zt}}{(z+1)^2(z+2)}\right) + \mathrm{Res}_{z=-2}\left(\frac{e^{zt}}{(z+1)^2(z+2)}\right)\Big).$$

[1] As we saw in class, when computing a Laplace transform, we always close the loop by using a semi-circle in the left direction, since, for $t \geqslant 0$, the complex function e^{tz} is bounded on half planes of the form $\mathrm{Re}(z) < \gamma$.

Adding the residues, we therefore obtain:

$$f(t) = e^{-2t} + e^{-t}(t-1) = \boxed{e^{-t}(t-1) + e^{-2t}}.$$

(b) $F(z) = \dfrac{z^2}{(z^2+1)^2}$

The singularities of F are at $z = \pm i$. Therefore, for any $\gamma > \mathrm{Re}(\pm i) = 0$, we compute using the formula for the inverse Laplace transform:

$$f(t) = \mathcal{L}^{-1}\left\{\frac{z^2}{(z^2+1)^2}\right\}(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{zt}\cdot\frac{z^2}{(z^2+1)^2}\,dz \tag{1}$$

The integrand has second-order poles at $z = i$ and $z = -i$. Let us compute the associated residues, setting

$$g(z) = \frac{e^{zt}z^2}{(z-i)^2(z+i)^2}$$

- The residue at the pole of order 2 at $z = i$ is:

$$\mathrm{Res}_{z=i}\, f(z) = \frac{d}{dz}\left[\frac{e^{zt}z^2}{(z+i)^2}\right]_{z=i}$$

Let $h(z) = \dfrac{e^{zt}z^2}{(z+i)^2}$. Then:

$$h'(z) = \frac{(2ze^{zt} + tz^2e^{zt})(z+i)^2 - 2(z+i)e^{zt}z^2}{(z+i)^4}$$

Evaluating at $z = i$, we therefore obtain

$$\mathrm{Res}_{z=i}\, g(z) = \frac{4te^{it} - 4ie^{it}}{16} = \frac{e^{it}}{4}(t-i).$$

- Similarly, we have

$$\mathrm{Res}_{z=-i}\, g(z) = \frac{e^{-it}}{4}(t+i).$$

In order to compute the integral (1) along $\mathrm{Re}(z) = \gamma$, as we did before, we close the loop towards the left i.e. we close the line segment $t \to \gamma + it$ for $t \in [-R, R]$ with the half circle $\theta \to Re^{i\theta}$ for $\theta \in [\frac{\pi}{2}, \frac{3\pi}{2}]$ and send $R \to +\infty$. This loop will contain all the poles of $e^{tz}F(z)$ and can be computed using the residue theorem. The integral over the half circle will go to 0 as $R \to +\infty$, leaving us with:

$$\int_{\gamma-i\infty}^{\gamma+i\infty} e^{zt}\frac{z^2}{({}^2{+}1)^2}\,dz = 2\pi i\Big(\mathrm{Res}_{z=i}\,(g(z)) + \mathrm{Res}_{z=-i}\,(g(z))\Big).$$

Thus,

$$f(t) = \mathcal{L}^{-1}\left\{\frac{z^2}{(z^2+1)^2}\right\}(t) = \left[\frac{e^{it}}{4}(t-i) + \frac{e^{-it}}{4}(t+i)\right] = \mathrm{Re}\left[\frac{e^{it}}{2}(t-i)\right]$$

Now use:

$$e^{it} = \cos t + i\sin t$$

Then:

$$(t-i)(\cos t + i\sin t) = t\cos t + it\sin t - i\cos t + \sin t = (t\cos t + \sin t) + i(t\sin t - \cos t)$$

Take the real part:

$$\boxed{f(t) = \frac{1}{2}\left(t\cos t + \sin t\right)}$$

5. (a) $f(t) = 0$

Take the Laplace transform of both sides. Let $Y(s) = \mathcal{L}\{y(t)\}(s)$. Use the Laplace transforms:

$$\mathcal{L}\{y''\} = s^2Y(s) - sy(0) - y'(0), \quad \mathcal{L}\{y'\} = sY(s) - y(0), \quad \mathcal{L}\{y\} = Y(s)$$

Substitute into the ODE:

$$s^2Y(s) - s - 1 + 2(sY(s) - 1) + Y(s) = 0$$

$$(s^2 + 2s + 1)Y(s) - (s + 2 + 1) = 0 \quad \Rightarrow \quad (s+1)^2Y(s) = s + 3$$

So:

$$Y(s) = \frac{s+3}{(s+1)^2}$$

Now simplify:

$$Y(s) = \frac{s+1+2}{(s+1)^2} = \frac{s+1}{(s+1)^2} + \frac{2}{(s+1)^2} = \frac{1}{s+1} + \frac{2}{(s+1)^2}$$

Now take the inverse Laplace transform:

$$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}(t) + 2\cdot\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\}(t) = e^{-t} + 2te^{-t}$$

(b) $f(t) = t$

Take Laplace transform of both sides:

$$(s^2 + 2s + 1)Y(s) = \mathcal{L}\{t\}(s) + (s + 3)$$

Recall:

$$\mathcal{L}\{t\} = \frac{1}{s^2}, \quad \text{so:} \quad (s + 1)^2 Y(s) = \frac{1}{s^2} + s + 3$$

Thus:

$$Y(s) = \frac{1}{s^2(s + 1)^2} + \frac{s + 3}{(s + 1)^2}$$

We already found:

$$\frac{s + 3}{(s + 1)^2} = \frac{1}{s + 1} + \frac{2}{(s + 1)^2}$$

Now compute inverse Laplace of:

$$\frac{1}{s^2(s + 1)^2}$$

Use known inverse (from tables or convolution), or use partial fractions:

Let:

$$\frac{1}{s^2(s + 1)^2} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s + 1} + \frac{D}{(s + 1)^2}$$

Multiply both sides by $s^2(s + 1)^2$, and solve for constants:

$$\frac{1}{s^2(s + 1)^2} = -\frac{2}{s} + \frac{1}{s^2} + \frac{2}{s + 1} + \frac{1}{(z + 1)^2}.$$

Final inverse transform is known:

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2(s + 1)^2}\right\}(t) = -2 + t + (2 + t)e^{-t}$$

Thus:

$$y(t) = \left[-2 + t + (2 + t)e^{-t}\right] + e^{-t} + 2te^{-t} == -2 + t + 3(1 + t)e^{-t}$$

6. Take the Laplace transform of both sides. Let $Y(s) = \mathcal{L}\{y(t)\}$. Recall the transforms:

$$\mathcal{L}\{y'(t)\} = sY(s) - y(0), \quad \mathcal{L}\{y''(t)\} = s^2Y(s) - sy(0) - y'(0), \quad \mathcal{L}\{y'''(t)\} = s^3Y(s) - s^2y(0) - sy'(0) - y''(0)$$

Substitute into the equation:

$$s^3Y(s) - s^2 \cdot 0 - s \cdot 0 - 1 + sY(s) - 0 = \mathcal{L}\{te^{-t}\} \Rightarrow (s^3 + s)Y(s) - 1 = \frac{1}{(s + 1)^2}$$

Solve for $Y(s)$:

$$(s^3+s)Y(s) = \frac{1}{(s+1)^2} + 1 \Rightarrow Y(s) = \frac{1}{(s+1)^2 \cdot s(s^2+1)} + \frac{1}{s(s^2+1)}$$

We now compute the inverse Laplace transform of both terms.

Second term:

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\}(t) = \int_0^t \sin(\tau)\, d\tau = 1 - \cos t$$

First term: Let us define:

$$f(t) = te^{-t}, \quad \mathcal{L}\{f(t)\} = \frac{1}{(s+1)^2}$$

$$g(t) = 1 - \cos t, \quad \mathcal{L}\{g(t)\} = \frac{1}{s(s^2+1)}$$

Then, using the convolution theorem:

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2 \cdot s(s^2+1)}\right\}(t) = \int_0^t f(t-\tau)g(\tau)\, d\tau = \int_0^t (t-\tau)e^{-(t-\tau)}(1-\cos\tau)\, d\tau = -\frac{1}{2}(t+2)e^{-t} - \frac{1}{2}(\sin(t$$

Therefore,

$$\boxed{y(t) = -\frac{1}{2}(t+2)e^{-t} - \frac{1}{2}\sin(t) - \cos(t) + 2}$$